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RELAXED SHAPE OPTIMIZATION: THE CASE OF NONNEGATIVE DATA FOR THE DIRICHLET PROBLEM

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0. INTRODUCTION

The aim of this paper is to discuss some qualitative properties of the relaxed solutions of a shape optimization problem for the domain of resolution of an elliptic equation with Dirichlet boundary conditions.

Given a bounded open subset Ω of $\mathbf{R}^n$, $n \geq 2$, and $f \in H^{-1}(\Omega)$, for any open subset A of Ω let u_A be the solution of the Dirichlet problem

$$u_A \in H_0^1(A) \ , \ -\Delta u_A = f \text{ in } A$$

extended by 0 outside A. Given a Carathéodory function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, we shall consider the optimal design problem

$$\inf_A \int_\Omega g(x, u_A(x)) \, dx \tag{0.1}$$

where A runs over all open subsets of Ω.

It can be shown (see [B.DM]) that the infimum in (0.1) is not always achieved. Therefore, in order to study the behaviour of the minimizing sequences of (0.1), a relaxed formulation of this problem has been introduced in [B.DM.], based on a new family of partial differential equations depending on measures. These equations, called "relaxed Dirichlet problems" in [DM.M.1] and [DM.M.2], can be written formally as

$$-\Delta u + \mu u = f \text{ in } \Omega \quad , \quad u = 0 \text{ on } \partial\Omega, \tag{0.2}$$

where μ is a nonnegative Borel measure on Ω which vanishes on all sets of (harmonic) capacity zero. Let us denote by $\mathcal{M}_0(\Omega)$ the class of all these measures.

We say that u is a solution to (0.2) if $u \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and if

$$\int_\Omega \nabla u \nabla \varphi \, dx + \int_\Omega u \, \varphi \, d\mu \; = < f, \varphi > \tag{0.3}$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$, where $<\, ,\, >$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It can be proved (see [DM.M.1]) that for every $\mu \in \mathcal{M}_0(\Omega)$ there exists one and only one solution u of (0.2). We will denote it by u_μ.

The relaxed formulation of (0.1) is the minimum problem

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x)) \, dx. \tag{0.4}$$

It can be shown (see [A.], [DM.M.2], [B.DM.]) that, under some natural growth conditions on g, we have

$$\inf_A \int_\Omega g(x, u_A(x)) \, dx = \min_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x)) \, dx. \tag{0.5}$$

Moreover, $\mu \in \mathcal{M}_0(\Omega)$ is a minimizer of (0.4) if and only if there exists a minimizing sequence (A_h) of (0.1) such that (u_{A_h}) converges to u_μ weakly in $H_0^1(\Omega)$. Finally, the original problem (0.1) admits a solution A if and only if the measure $\mu_A \in \mathcal{M}_0(\Omega)$ defined by

$$\mu_A(B) = \begin{cases} 0 & \text{if } \operatorname{Cap}(B \backslash A) = 0, \\ \\ +\infty & \text{if } \operatorname{Cap}(B \backslash A) > 0. \end{cases} \tag{0.6}$$

is a minimizer of the relaxed problem (0.4).

In this paper we study the case where $f \geq 0$ in Ω. Then (0.4) can be rewritten as a minimization problem on some closed convex set of $H_0^1(\Omega)$ for which it is easy to show the existence of at least one minimizer. This will be proved in section 1. In section 2 we will study more general properties of the solutions. In section 3 we will consider the particular case where $g(x, .)$ is differentiable and we will characterize the minimizers of the relaxed problem (0.4) by means of a complementarity system (Theorem 7). In section 4 we will analyse the shape of the solutions u_μ of (0.4) in the case where Ω is a ball and the data are nonnegative and radially symmetric. In this case we are able to give a fairly accurate description of the solutions. In particular the nature of the measure μ which minimizes (0.4) will be investigated.

Our analysis shows that, under very general assumptions (see Theorems 2 and 9), all the minimizers μ of the relaxed problem (0.4) are Radon measures. This implies that the measure μ_A defined by (0.6) cannot be a minimizer of (0.4) unless $A = \Omega$. This leads to the surprising conclusion that, under these hypotheses, the shape optimization problem (0.1) cannot have a minimizer A unless $A = \Omega$.

We will conclude the paper with the discussion of an example where the optimal measure μ for (0.4) and the corresponding minimizing sequence A_h for (0.1) can be described explicitly.

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1. FORMULATION OF THE PROBLEM

Let Ω be a bounded, open subset of $\mathbf{R}^n$, $n \geq 2$.

We will denote by $H^1(\Omega)$, $H_0^1(\Omega)$ the usual Sobolev spaces constructed on $L^2(\Omega)$. $H^{-1}(\Omega)$ will be the dual space of $H_0^1(\Omega)$, $< \, , \, >$ the duality bracket between these two spaces (for details, see, for instance, [K.S.], [G.T.]).

On $H_0^1(\Omega)$ we will find convenient to use the norm

$$||\nabla u||_2 = (\int_\Omega |\nabla u|^2 dx)^{1/2} \tag{1.1}$$

where $|\nabla u|$ denotes the usual Euclidean norm of the gradient of u, and $| \ \ |_2$ the usual L^2 norm on $L^2(\Omega)$. $| \ \ |_{H^{-1}(\Omega)}$ will be the dual norm of (1.1) on $H^{-1}(\Omega)$.

For every compact subset E of Ω we define the capacity of E by

$$\mathrm{Cap}(E) = Inf \ \{\int_\Omega |\nabla u|^2 dx \quad : \quad u \in C_0^1(\Omega) \ , \ u \geq 1 \quad \text{on } E\}.$$

This definition is extended to open subsets $O \subset \Omega$ by

$$\mathrm{Cap}(O) = Sup \ \{\mathrm{Cap}(K) \ : \ K \text{ compact}, \ K \subset O\}$$

and to arbitrary sets $E \subset \Omega$ by

$$\mathrm{Cap}(E) = Inf \ \{\mathrm{Cap}(O) \ : \ O \text{ open}, \ E \subset O\}.$$

"a.e." means "up to a set of measure zero", "q.e." means "up to a set of capacity zero".

$\mathcal{B}(\Omega)$ denotes the σ-algebra of Borel subsets of Ω.

By "measure" we mean a countably additive set function such that $\mu(\emptyset) = 0$ ($\emptyset$ is the empty set). The Lebesgue measure is denoted by dx or alternatively by $meas$.

$\mathcal{M}_0(\Omega)$ is the set of all measures $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ such that (see [DM.M.1], [DM.M.2], [B.DM.]) :

$$\mu(B) = 0 \text{ for every } B \in \mathcal{B}(\Omega) \text{ with } \mathrm{Cap}(B) = 0.$$

If $u \in H^1(\Omega)$, we always assume that the pointwise value $u(x)$ of u at each point x of Ω satisfies

$$\liminf_{\rho \to 0+} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} u(y)\, dy \leq \ u(x) \ \leq \limsup_{\rho \to 0+} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} u(y)\, dy \tag{1.2}$$

where $B(x,\rho) = \{y \in \Omega \mid |y-x| < \rho\}$ and $|B(x,\rho)|$ the Lebesgue measure of $B(x,\rho)$. With this convention the pointwise value $u(x)$ is uniquely determined q.e. in Ω and the function u is quasi continuous in Ω (see [F.Z.]). Therefore, the integrals which appear in the definition (0.3) of a solution of (0.2) are defined unambiguously.

If $\nu \in H^{-1}(\Omega)$, and $\nu \geq 0$ in the sense of distributions, then there exists a nonnegative Radon measure on Ω, still denoted by ν, such that $\nu \in \mathcal{M}_0(\Omega)$, and for every $u \in H_0^1(\Omega)$ we have

$$u \in L^1(\Omega, \nu) \quad , \quad <\nu, u> = \int_\Omega u \, d\nu, \tag{1.3}$$

where the pointwise values of u are determined according to our convention (1.2).

Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be such that

$$g(x,s) \ \text{ is a Carathéodory function} \tag{1.4}$$

i.e. measurable in x for every $s \in \mathbf{R}$ and continuous in s for almost every $x \in \Omega$. Moreover, assume that there exist a function $a_0 \in L^1(\Omega)$ and a constant b_0 such that

$$|g(x,s)| \leq a_0(x) + b_0|s|^2 \quad \text{for a.e. } \ x \in \Omega, \forall s \in \mathbf{R}. \tag{1.5}$$

Let f be in $H^{-1}(\Omega)$. All along the paper we will assume that

$$f \geq 0 \qquad \text{in} \qquad \Omega. \tag{1.6}$$

Here the sign "$\geq$" is understood either in the sense of distributions, measures, or $H^{-1}(\Omega)$, since all these meanings are equivalent by (1.3). We recall that for every $\mu \in \mathcal{M}_0(\Omega)$ the function u_μ is the unique solution of the relaxed Dirichlet problem (0.2) according to the definition (0.3).

We will consider the problem of minimizing the integral :

$$\int_\Omega g(x, u_\mu(x))\, dx$$

over all the measures $\mu \in \mathcal{M}_0(\Omega)$. We will denote in short this problem by

$$\inf_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x))\, dx \tag{1.7}$$

The relationship between (1.7) and the optimal design problem (0.1) has been discussed in the introduction.

We now want to give an equivalent formulation of problem (1.7) in terms of a minimization problem on a closed convex subset of $H_0^1(\Omega)$.

Set

$$K = \{u \in H_0^1(\Omega) \ : \ \Delta u + f \geq 0 \text{ in } \Omega\} \tag{1.8}$$

$$K^+ = \{u \in K \ : \ u \geq 0 \ \text{ q.e. in } \Omega\}. \tag{1.9}$$

Again, in the definition of K the sign "$\geq$" is understood either in the sense of distributions, measures, or $H^{-1}(\Omega)$. K is the set of all subsolutions to the equation

$$-\Delta u = f$$

which belong to $H_0^1(\Omega)$. In particular one has (see [K.S.])

$$u, v \in K \qquad \Rightarrow \qquad u \vee v \in K. \tag{1.10}$$

($u \vee v$ denotes the function $Max(u, v)$, similarly $u \wedge v$ denotes $Min(u, v)$). It is easy to check that K is a closed convex subset of $H_0^1(\Omega)$. Moreover, we have :

Proposition 1 : Assume that $f \in H^{-1}(\Omega)$ satisfies (1.6), then K^+ is a closed, bounded, convex subset of $H_0^1(\Omega)$ and, if (1.4), (1.5) hold, then the problem

$$\inf_{u \in K^+} \int_\Omega g(x, u(x)) \, dx \tag{1.11}$$

admits at least one minimizer.

Proof : The fact that K^+ is closed and convex is easy to check. To see that K^+ is bounded, note that, if $u \in K^+$, then

$$< \Delta u + f, u > \ \geq \ 0.$$

This reads also

$$\int_\Omega |\nabla u|^2 \, dx \leq \ < f, u > \ \leq |f|_{H^{-1}(\Omega)} \, ||\nabla u||_2$$

and the boundedness of K^+ follows (see (1.1)).

Next, due to (1.4), (1.5),

$$u \longrightarrow \int_\Omega g(x, u(x)) \, dx$$

is continuous from $L^2(\Omega)$ into $\mathbf{R}$ (see for instance [N]) and thus it is also sequentially continuous from $H_0^1(\Omega)$-weak into $\mathbf{R}$ (this is due to the complete continuity of the canonical embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$). Then, the result follows from the fact that the closed, convex, bounded set K^+ is sequentially weakly compact.

We have the following characterization for the elements of K^+ :

Theorem 1 : Let u be in $H_0^1(\Omega)$. Then, $u \in K^+$ if and only if $u = u_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$.

Proof : First, let u be an element of K^+. Then, $\nu = \Delta u + f$ is a nonnegative Radon measure which belongs to $H^{-1}(\Omega)$ and satisfies (see (1.3)) :

$$\int_\Omega \varphi \; d\nu = <\nu, \varphi> = -\int_\Omega \nabla u \nabla \varphi \; dx + <f, \varphi> \tag{1.12}$$

for every $\varphi \in H_0^1(\Omega)$.

Now, clearly, from (1.3) we see that $u \in L^1(\Omega, d\nu)$ and from (1.12) we get

$$\int_\Omega u \; d\nu = -\int_\Omega |\nabla u|^2 \; dx + <f, u> . \tag{1.13}$$

In particular u is $d\nu$ measurable and since $u \geq 0$ we can define $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ by :

$$\mu(B) = \begin{cases} \int_B \frac{1}{u} \; d\nu & \text{if } \operatorname{Cap}(B \cap \{u = 0\}) = 0, \\ \\ +\infty & \text{if } \operatorname{Cap}(B \cap \{u = 0\}) > 0. \end{cases} \tag{1.14}$$

$\{u = 0\}$ denotes the set $\{x \in \Omega \; : \; u(x) = 0\}$. Similar notation will be used without any further notice for $\{u > 0\}$, $\{u \geq 0\}$,...

If $B \in \mathcal{B}(\Omega)$ and $\operatorname{Cap}(B) = 0$, then it is easy to show that $\nu(B) = 0$ and thus $\mu(B) = 0$.

Since $u \in L^1(\Omega, d\nu)$, u is μ-measurable. Moreover by (1.13) and (1.14) we have

$$\int_\Omega u^2 \; d\mu = \int_{\{u>0\}} u^2 \; d\mu = \int_{\{u>0\}} u \; d\nu = \int_\Omega u \; d\nu = -\int_\Omega |\nabla u|^2 \; dx + <f, u> < +\infty,$$

hence $u \in L^2(\Omega, d\mu)$.

Let $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, d\mu)$. By (1.14), we have $\varphi = 0$ q.e. on $\{u = 0\}$. Therefore, (1.12) gives

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi \; dx + \int_\Omega u\varphi \; d\mu &= \int_\Omega \nabla u \nabla \varphi \; dx + \int_{\{u>0\}} u\varphi \; d\mu \\ &= \int_\Omega \nabla u \nabla \varphi \; dx + \int_{\{u>0\}} \varphi \; d\nu \; = <f, \varphi> . \end{aligned}$$

This ends the proof of the fact that $u = u_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$.

Conversely, assume now that $u = u_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$. It is shown in [DM.M.1], Proposition 2.9, that $u \geq 0$ q.e. in Ω. Moreover, using the approximation procedure of [DM.M.1], Proposition 2.6, we easily show that

$$\int_\Omega \nabla u \nabla \varphi \; dx - <f, \varphi> \leq 0 \qquad \forall \varphi \in H_0^1(\Omega), \;\; \varphi \geq 0 \;\; \text{a.e. in } \Omega.$$

Thus,

$$\Delta u + f \geq 0$$

in the sense of $H^{-1}(\Omega)$ and $u \in K^+$.

Remark 1 : In general the measure μ that satisfies (0.2) is not uniquely determined by f and u. For instance, when $f = 0$ and $u = 0$, then any measure $\mu \in \mathcal{M}_0(\Omega)$ is suitable. However, we have :

Proposition 2: Let $u = u_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$. Then $\nu = \Delta u + f$ is a nonnegative Radon measure in $H^{-1}(\Omega)$. Moreover,

$$\nu(B) = \int_B u \, d\mu \quad \forall B \in \mathcal{B}(\Omega), \quad \text{Cap } (B \cap \{u = 0\}) = 0. \tag{1.15}$$

If $u > 0$ q.e. in Ω, then

$$\nu = u\mu \qquad , \qquad \mu = \frac{\nu}{u} \tag{1.16}$$

as measures in Ω. So, in this case, the measure μ such that $u = u_\mu$ is uniquely determined by f and u.

Proof : The fact that ν is a Radon measure has already been proved in Theorem 1 and (1.15), (1.16) are clear from (1.14).

Remark 2 : If $u > 0$ q.e. in Ω and $1/u \in L^\infty_{loc}(\Omega, d\mu)$, then μ is a Radon measure. If, in addition, $1/u \in L^\infty(\Omega, d\mu)$, then μ belongs to $H^{-1}(\Omega)$. To verify this last assertion, denote by k a constant such that

$$0 \le \frac{1}{u} \le k \quad \mu - \text{a.e. in } \Omega.$$

Then, for every $\varphi \in \mathcal{D}(\Omega)$,

$$|<\mu, \varphi>| \le \int_\Omega \frac{|\varphi|}{u} \, d\nu \ \le k < \nu, |\varphi| > \le k \ |\nu|_{H^{-1}(\Omega)} \ ||\nabla\varphi||_2$$

which yields $\mu \in H^{-1}(\Omega)$.

As a consequence of Theorem 1 we have :

Corollary 1 : The problems

$$\inf_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x)) \, dx \tag{1.17}$$

$$\inf_{u \in K^+} \int_\Omega g(x, u(x)) \, dx \tag{1.18}$$

are equivalent in the sense that, if μ is a measure minimizer of (1.17), then $u_\mu \in K^+$ and u_μ is a minimizer of (1.18); conversely, if u is a minimizer of (1.18) on K^+, then there exists $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ and μ is a minimizer for (1.17).

Proof : This is an immediate consequence of Theorem 1 which claims that

$$\{u_\mu \mid \mu \in \mathcal{M}_0(\Omega)\} = K^+$$

Remark 3 : In the case where $g(x,.)$ is strictly convex for a.e. x in Ω, then the solution u to (1.18) is of course unique. Moreover, if $u > 0$ q.e. on Ω, then it results from Proposition 2 that the measure μ minimizing (1.17) is unique too.

One can also try to minimize

$$\int_\Omega g(x,u(x))\,dx \tag{1.19}$$

over K. In this case one has :

Proposition 3 : If

$$g(x,s) > g(x,0) \quad \text{for a.e.} \ \ x \in \Omega, \ \forall \ \ s < 0$$

then (1.18) is equivalent to

$$\inf_{u \in K} \int_\Omega g(x,u(x))\,dx \tag{1.20}$$

(i.e. these two problems have the same infimum and the same minimizers).

Proof: Since $f \geq 0$ implies that $0 \in K$, by (1.10) we have that $u^+ \in K^+$ for every $u \in K$. Therefore, we will be done if we can prove that

$$\int_\Omega g(x,u)\,dx > \int_\Omega g(x,u^+)\,dx$$

for every $u \in K \backslash K^+$. Let us fix $u \in K \backslash K^+$. We have $\text{meas}(\{u < 0\}) > 0$ and $g(x,u) > g(x,0)$ a.e. on $\{u < 0\}$. Hence,

$$\int_\Omega g(x,u(x))\,dx = \int_{\{u \geq 0\}} g(x,u(x))\,dx + \int_{\{u<0\}} g(x,u(x))\,dx \ >$$

$$\int_{\{u \geq 0\}} g(x,u(x))\,dx + \int_{\{u<0\}} g(x,0)\,dx = \int_\Omega g(x,u^+(x))\,dx.$$

The proof is complete.

2. SOME PROPERTIES OF THE SOLUTIONS

In this section we collect some properties of the solutions u to (1.18) or (1.20), under some special but mild assumptions on g. First of all, for $x \in \Omega$ introduce the function Ψ defined by

$$\Psi(x) = \mathrm{Sup}\{s \mid g(x,s) \text{ is decreasing on } (-\infty, s)\} \tag{2.1}$$

Theorem 2 : Assume that

$$f \geq 0 \quad , \quad f \not\equiv 0 \text{ in any connected component of } \Omega \tag{2.2}$$

$$inf_{\Omega'} \ \Psi = m_{\Omega'} > 0 \quad \forall \ \ \Omega' \subset\subset \Omega$$

($inf_{\Omega'}$ denotes the essential infimum on Ω' with respect to the Lebesgue measure). Let u be a minimizer of (1.18) or (1.20). Then

$$inf_{\Omega'} \ u = C_{\Omega'} > 0 \quad \forall \ \ \Omega' \subset\subset \Omega. \tag{2.3}$$

Therefore, the measure $\mu \in \mathcal{M}_0(\Omega)$ which satisfies $u = u_\mu$ is uniquely determined by u and is a Radon measure.

Remark 4 : In the case we are considering, $\Psi > 0$ a.e. in Ω, and thus $g(x,.)$ is decreasing for a.e. $x \in (-\infty, 0)$ and $g(x,s) > g(x,0)$ for a.e. x in Ω and for every $s < 0$. So, (1.18) and (1.20) have the same solutions.

Proof : Without loss of generality we can assume that Ω is connected. Set

$$K^\Psi = \{v \in H_0^1(\Omega) \ \mid \ v \leq \Psi \ \text{ a.e. in } \ \Omega\} \tag{2.4}$$

and introduce z the solution to

$$z \in K^\Psi \quad , \quad < -\Delta z - f, v - z > \geq 0 \quad \forall \, v \in K^\Psi \tag{2.5}$$

(K^Ψ is not empty since $0 \in K^\Psi$). Since $u \in K$, we have $\Delta u + f \geq 0$. Moreover, $\Delta z + f \geq 0$, so that $u \vee z \in K$.

Next, compute

$$\int_\Omega g(x, (u \vee z)(x)) \, dx = \int_{\{u<z\}} g(x, z(x)) \, dx + \int_{\{u \geq z\}} g(x, u(x)) \, dx$$

Since $z \leq \Psi$, if $\{u < z\}$ has a positive Lebesgue measure one would get by the definition of Ψ (see (2.1)):

$$\int_\Omega g(x,(u\vee z)(x))\,dx < \int_\Omega g(x,u(x))\,dx$$

and a contradiction with the definition of u. Thus $u \geq z$ a.e. in Ω.

We want to prove that $inf_{\Omega'}\ z > 0$ for every $\Omega' \subset\subset \Omega$. For this, fix $\Omega' \subset\subset \Omega$ and choose Ω'' connected and smooth such that

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega \quad \text{and} \quad f \not\equiv 0 \ \text{ in } \ \Omega''.$$

Let w be the solution to

$$w \in H^1_0(\Omega'') \quad , \quad -\Delta w = f \quad \text{in} \ \ \Omega''.$$

Since $w < +\infty$ q.e. in Ω'', there exists a constant $t > 0$ such that the measure f is not identically zero on $\{w \leq t\}$. Let χ be the characteristic function of $\{w \leq t\}$ and let θ be the solution to

$$\theta \in H^1_0(\Omega'') \quad , \quad -\Delta\theta = k\chi f \quad \text{in} \ \ \Omega'',$$

where $k = m_{\Omega''}/t$. Since $k\chi f \not\equiv 0$ in Ω'', by the strong maximum principle we have

$$inf_{\Omega'}\ \theta = C'_{\Omega'} > 0$$

Since $\chi f \leq f$, an easy comparison argument shows that $\theta \leq kw$ q.e. in Ω'', hence $\theta \leq m_{\Omega''}$ q.e. in $\{w \leq t\}$. We claim that $\theta \leq m_{\Omega''}$ q.e. in Ω''. This is a consequence of the Domination Principle of potential theory (see [D.], 1. V. 10). An alternative variational proof is the following : if we multiply by $(\theta - m_{\Omega''})^+$ both sides of the equation which defines θ, we get

$$\int_{\Omega''} |\nabla(\theta - m_{\Omega''})^+|^2\,dx = < -\Delta\theta, (\theta - m_{\Omega''})^+ >$$

$$= < k\chi f, (\theta - m_{\Omega''})^+ > = k\int_{\{w\leq t\}} (\theta - m_{\Omega''})^+\,df = 0,$$

hence $(\theta - m_{\Omega''})^+ = 0$ q.e. in Ω'', which implies $\theta \leq m_{\Omega''}$ q.e. in Ω''. Now, if we denote by $\tilde\theta$ the extension of θ by 0 outside Ω'', we easily get (see the Appendix, Lemma A)

$$-\Delta\tilde\theta \leq k\chi f \leq f \quad \text{in} \quad \Omega.$$

Since $\tilde\theta \leq m_{\Omega''} \leq \Psi$ a.e. in Ω'' and $\tilde\theta = 0 \leq \Psi$ a.e. in $\Omega\backslash\Omega''$, we have $\tilde\theta \in K^\Psi$. Since z is the largest subsolution of the equation $-\Delta v = f$ which belongs to K^Ψ (see [K.S.]), we obtain $z \geq \tilde\theta$ a.e. in Ω, hence

$$u \geq z \geq \tilde\theta \geq C'_{\Omega'} > 0 \quad \text{a.e. in} \quad \Omega'.$$

This concludes the proof of (2.3). The assertions regarding the measure μ follow from Proposition 2 and Remark 2.

Let us consider the solution u_0 to the problem

$$u_0 \in H_0^1(\Omega) \quad , \quad -\Delta u_0 = f \quad \text{in} \quad \Omega. \tag{2.6}$$

Remark 5 : The function u_0 is the solution of (0.2) corresponding to $\mu = 0$. Note also that u_0 is the largest element of K. This is a consequence of the maximum principle, since for any $u \in K$ one has $-\Delta u \leq f = -\Delta u_0$. In particular for any $\mu \in \mathcal{M}_0(\Omega)$ one has $u_\mu \leq u_0$ q.e. in Ω .

If E is a subset of Ω, the essential closure of E is the closed set

$$ess.\ cl\ E = \{x \in \Omega \ : \ meas(E \cap B(x,\rho)) > 0 \ \ \forall\, \rho > 0\}.$$

Then, if Supp ν denotes the support of a measure ν one has :

Theorem 3 : Assume (1.6). Let u be a minimizer of (1.18) or (1.20). Then the Radon measure

$$\nu = \Delta u + f$$

satisfies

$$\text{Supp}\ \nu \subset ess.\ cl\ \{u_0 \geq \Psi\}.$$

(We do not assume here necessarily that (1.18), (1.20) are equivalent, but (1.20) is assumed to have a minimizer).

Proof : Set $U = \Omega \backslash ess.\ cl\ \{u_0 \geq \Psi\}$. Then, U is open and $u_0 < \Psi$ a.e. in U. Let B be a ball included in U. Assume by contradiction that $\nu \not\equiv 0$ on B and introduce ξ the solution to

$$\xi \in H_0^1(B) \quad , \quad -\Delta \xi = \nu = \Delta u + f \quad \text{in} \quad B. \tag{2.7}$$

(This makes sense since $\nu \in H^{-1}(B)$). Then, it is clear that $\xi \geq 0$ and if $\nu \not\equiv 0$ on B we have $\xi > 0$ on B by the strong maximum principle. Denote by $\tilde{\xi}$ the extension of ξ by 0 outside B. Since $\xi \geq 0$ and

$$-\Delta \xi \leq \Delta u + f \quad \text{on} \quad B$$

one has (see the Appendix)

$$-\Delta \tilde{\xi} \leq \Delta u + f \quad \text{on} \quad \Omega,$$

hence,

$$u + \tilde{\xi} \in K^+ \text{ or } K$$

if $u \in K^+$ or K respectively. Next, since $u + \tilde{\xi} \in K$, one has $u + \tilde{\xi} \leq u_0$, hence $u + \xi \leq u_0 < \Psi$ a.e. on B.
Then, since $\xi > 0$ on B,

$$\int_\Omega g(x, u + \tilde{\xi})\ dx = \int_B g(x, u + \xi)\ dx + \int_{\Omega \backslash B} g(x,u)\ dx < \int_\Omega g(x,u)\ dx$$

and a contradiction with the definition of u. Thus $\nu \equiv 0$ on B, and since this holds for any B the result follows.

In the same spirit, one has :

Theorem 4 : Assume that u is a minimizer of (1.18) or (1.20). Then, for any $\epsilon > 0$, the Radon measure

$$\nu = \Delta u + f$$

satisfies

$$\text{Supp } \nu \subset ess.cl. \ \{u \geq \Psi - \epsilon\}.$$

Proof : Set $U_\epsilon = \Omega \backslash ess.cl. \ \{u \geq \Psi - \epsilon\}$. Then U_ϵ is open and $u < \Psi - \epsilon$ a.e. in U_ϵ. Let B be a ball included in U_ϵ. We want to prove that $\nu \equiv 0$ on B. If not, introduce the solution w to

$$w \in H_0^1(B) \quad , \quad -\Delta w = \nu \quad \text{in} \quad B.$$

Since $w < +\infty$ q.e. in B and $\nu \not\equiv 0$ in B, there exists a constant $t > 0$ such that $\nu \not\equiv 0$ in $\{w \leq t\}$. Let χ be the characteristic function of $\{w \leq t\}$ and let ξ be the solution to

$$\xi \in H_0^1(B) \quad , \quad -\Delta \xi = \chi \nu \quad \text{in} \quad B. \tag{2.8}$$

Since $\chi\nu \geq 0$ and $\chi\nu \not\equiv 0$ in B, by the strong maximum principle we have $\xi > 0$ in B. Since $\chi\nu \leq \nu$, we have also $\xi \leq w$ q.e. in B, hence, in particular, $\xi \leq t$ q.e. in $\{w \leq t\}$. Arguing as in the proof of Theorem 2 we obtain $\xi \leq t$ q.e. in B.

Since $-\Delta\xi \geq 0$, for any $0 < \delta < 1$ one has

$$-\delta\Delta\xi \leq -\Delta\xi \leq \nu = \Delta u + f \quad \text{in} \quad B. \tag{2.9}$$

If $\tilde{\xi}$ denotes the extension of ξ by 0 outside B, then, clearly, for δ small enough, (i.e. $0 < \delta < min(1, \epsilon/t)$), one has

$$u + \delta\tilde{\xi} \in K^+ \text{ or } K \quad , \quad u + \delta\tilde{\xi} < \Psi \quad \text{a.e. on} \quad B \tag{2.10}$$

(see the Appendix, Lemma A). Thus, if ν does not vanish identically on B, one has $\xi > 0$ on B and

$$\int_\Omega g(x, u + \delta\tilde{\xi}) \, dx = \int_B g(x, u + \delta\xi) \, dx + \int_{\Omega\backslash B} g(x, u) \, dx < \int_\Omega g(x, u) \, dx$$

and a contradiction with the definition of u. Thus, $\nu \equiv 0$ on B and this ends the proof of the theorem.

As a consequence we have

Corollary 2 : Assume that u is a minimizer of (1.18) or (1.20). Then, if $u - \Psi$ is continuous, the Radon measure

$$\nu = \Delta u + f$$

satisfies

$$\text{Supp } \nu \subset \ \{u \geq \Psi\}.$$

Proof : This follows from the fact that

$$ess.cl. \ \{u \geq \Psi - \epsilon\} \subset \{u \geq \Psi - \epsilon\}.$$

Thus

$$\text{Supp } \nu \subset \cap_{\epsilon > 0} \ \{u \geq \Psi - \epsilon\} = \{u \geq \Psi\}.$$

Let us now get some information on the set $\{u < \Psi\}$. For that, let us first prove :

Proposition 4 : Assume $u, \hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega)$,

$$\Delta\hat{\Psi} + f \geq 0 \ \text{ in } \Omega \quad , \qquad \Delta u + f \geq 0 \ \text{ in } \Omega. \tag{2.11}$$

Let C be a connected component of $\{u < \hat{\Psi}\}$ and let $z = u + \chi_C(\hat{\Psi} - u)$, where χ_C is the characteristic function of C. Then

$$z \in H^1(\Omega) \quad , \qquad \Delta z + f \geq 0 \ \text{ in } \Omega. \tag{2.12}$$

($C^0(\Omega)$ is the space of continuous functions in Ω).

Proof : Let us prove that $z \in H^1(\Omega)$. For every $\epsilon > 0$ and for every $\Omega' \subset\subset \Omega$ let

$$C_\epsilon = \{x \in C \cap \Omega' \ : \ \hat{\Psi}(x) > u(x) + \epsilon\}.$$

Since $C_\epsilon \subset\subset C$, there exists $\varphi_\epsilon \in C_0^1(\Omega)$ such that $0 \leq \varphi_\epsilon \leq 1$ in Ω, $\varphi_\epsilon = 1$ in C_ϵ, and $\varphi_\epsilon = 0$ outside C.

Then the function

$$z_\epsilon = [(1 - \varphi_\epsilon)u + \varphi_\epsilon(\hat{\Psi} - \epsilon)] \vee u$$

belongs to $H^1(\Omega)$ (see [K.S.]). Using the fact that $z_\epsilon = u$ in $\Omega' \backslash C_\epsilon$ and $z_\epsilon = \hat{\Psi} - \epsilon$ in C_ϵ, we obtain that z_ϵ is bounded in $H^1(\Omega')$. Since

$$z_\epsilon \to z \quad \text{in} \quad L^2(\Omega') \ \text{ as } \ \epsilon \to 0,$$

we obtain $z \in H^1(\Omega')$, hence $z \in H^1_{loc}(\Omega)$. Since $\nabla z = \nabla u$ a.e. in $\Omega \backslash C$ and $\nabla z = \nabla\hat{\Psi}$ a.e. in C, we conclude that $\nabla z \in L^2(\Omega)$, hence $z \in H^1(\Omega)$.

Next, consider w the solution to

$$w \in K^z = \{v \in H^1(\Omega) \mid v \leq z \ \ a.e. \ \text{in} \ \ \Omega \ , \ v - z \in H_0^1(\Omega)\}$$

$$< -\Delta w - f, v - w > \ \geq \ 0 \quad \forall v \in K^z. \tag{2.13}$$

First, we have $u \leq w$.

Indeed, plugging $v = w + (u - w)^+$, which belongs to K^z, in (2.13), we get

$$< -\Delta w - f, (u - w)^+ > \ \geq \ 0.$$

Subtracting this to

$$< -\Delta u - f, (u - w)^+ > \ \leq \ 0$$

(note that $(u - w)^+ \in H_0^1(\Omega)$) we obtain

$$< -\Delta(u - w), (u - w)^+ > \ \leq \ 0$$

and thus $u \leq w$.

Next, we prove that $z = w$.

Indeed, taking $v = z$ in (2.13) and using the inequality $w \leq z$ together with (2.11) we get

$$< -\Delta w - f, z - w > \ \geq \ 0$$

$$< -\Delta \hat{\Psi} - f, z - w > \ \leq \ 0.$$

Subtracting the first equation from the second one we obtain

$$< -\Delta(\hat{\Psi} - w), z - w > \ \leq \ 0,$$

which reads also, since $u \leq w \leq z$,

$$< -\Delta(z - w), z - w > \ \leq \ 0.$$

Hence $z = w$. Since $\Delta w + f \geq 0$, the result follows.

A consequence of this proposition is :

Theorem 5 : Assume that $\hat{\Psi}$ is a function satisfying $\hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega)$, $\Delta\hat{\Psi} + f \geq 0$ in Ω, and

$$g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s < \hat{\Psi}(x), \ \text{ for a.e. } \ x \in \Omega. \tag{2.14}$$

Let $u \in C^0(\Omega)$ be a minimizer of (1.18) or (1.20). Then the set $\{u < \hat{\Psi}\}$ cannot have a connected component C such that $C \subset\subset \Omega$.

Proof: Set $z = u + \chi_C(\hat{\Psi} - u)$. Then, it results from the previous proposition that $z \in K^+$ or K. Moreover,

$$\int_\Omega g(x,z)\ dx = \int_C g(x,\hat{\Psi})\ dx + \int_{\Omega\setminus C} g(x,u)\ dx < \int_\Omega g(x,u)\ dx.$$

This contradicts the fact that u is a minimizer and the result follows.

Remark 6 : The function Ψ defined by (2.1) could be a particular $\hat{\Psi}$. However, as we just saw, properties like (2.14) are relevant here and are weaker than to impose, for instance, that $g(x,.)$ is decreasing. We denote by $\hat{\Psi}$ functions for which we assume a property of this type leaving the notation Ψ for the function defined by (2.1).

An other consequence is :

Theorem 6 : Assume that $\hat{\Psi}$ is a function satisfying $\hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega)$, $\hat{\Psi} \geq 0$, $\Delta\hat{\Psi} + f \geq 0$ in Ω, and

$$g(x,s) > g(x,\hat{\Psi}(x)) \quad \forall s > \hat{\Psi}(x), \quad \text{for a.e. } x \in \Omega. \tag{2.15}$$

Let $u \in C^0(\Omega)$ be a solution to (1.18) or (1.20). Then the set $\{u > \hat{\Psi}\}$ cannot have a connected component C such that

$$C \subset\subset \{u \geq \hat{\Psi}\}^0. \tag{2.16}$$

(E^0 denotes the interior of the set E).

Proof : Assume that C is a connected component of $\{u > \hat{\Psi}\}$ such that (2.16) holds. Set

$$\tilde{u} = \begin{cases} \hat{\Psi}(x) & \text{if } x \in C \\ u(x) & \text{if } x \in \Omega\setminus C. \end{cases}$$

On $\{u \geq \hat{\Psi}\}^0$ one has

$$\tilde{u} = \vee_i \quad \{\hat{\Psi} + \chi_{C_i}(u - \hat{\Psi})\}$$

where C_i denote the connected components of $\{u > \hat{\Psi}\}$ distinct from C. Thus, (see (2.12))

$$\Delta\tilde{u} + f \geq 0 \quad \text{on} \quad \{u \geq \hat{\Psi}\}^0.$$

Moreover, if we denote by $\overline{C}$ the closure of C in Ω, on $\Omega\setminus\overline{C}$ one has

$$\Delta\tilde{u} + f = \Delta u + f \geq 0.$$

Since the two open sets $\Omega\setminus\overline{C}$, $\{u \geq \hat{\Psi}\}^0$ cover Ω, we have

$$\Delta\tilde{u} + f \geq 0 \quad \text{in} \quad \Omega.$$

Then, it is clear that $\tilde{u} \in K^+$ or K. So, we get

$$\int_\Omega g(x,\tilde{u})\,dx = \int_C g(x,\hat{\Psi})\,dx + \int_{\Omega\setminus C} g(x,u)\,dx < \int_\Omega g(x,u)\,dx.$$

This contradicts the fact that u is a minimizer and the result follows.

3. THE ADJOINT PROBLEM

In this section we assume that $g(x,.)$ is smooth for a.e. $x \in \Omega$. More precisely, we assume that this function is differentiable for a.e. $x \in \Omega$ and that the derivative

$$g_s(x,s) \quad \text{is a Carathéodory function.} \tag{3.1}$$

Moreover, we assume that there exist $a_1 \in L^2(\Omega)$, and a constant b_1 for which

$$|g_s(x,s)| \leq a_1(x) + b_1|s| \quad \text{for a.e. } x \in \Omega \quad , \quad \forall s \in \mathbf{R}. \tag{3.2}$$

In this case, (see [B. DM.]), when u denotes a minimizer of (1.20), it is useful to introduce the solution v of the so called adjoint problem :

$$v \in H_0^1(\Omega) \quad , \quad -\Delta v = g_s(x,u) \quad \text{in} \quad \Omega. \tag{3.3}$$

One has :

Theorem 7 : Assume (1.4), (1.5), (1.6), (3.1), (3.2). If u is a minimizer to (1.20) and v is the solution to (3.3), then (u,v) satisfies the complementarity problem

$$\Delta u + f \geq 0 \quad , \quad v \leq 0 \quad \text{q.e. in } \Omega \quad , \quad <\Delta u + f, v>= 0. \tag{3.4}$$

Conversely, if $(u,v) \in (H_0^1(\Omega))^2$ is a solution to (3.4) and

$$g(x,.) \text{ is convex for a.e. } x \in \Omega, \tag{3.5}$$

then u is a minimizer to (1.20).

Proof : If u is a minimizer of (1.20) on K, then for any $t \in (0,1)$ one has

$$\int_\Omega [g(x,u+t(w-u)) - g(x,u)]\,dx \geq 0 \quad \forall w \in K.$$

Dividing by t and letting $t \to 0$ we get

$$\int_\Omega g_s(x,u).(w-u)\ dx \geq 0 \quad \forall w \in K.$$

Hence, by (3.3),

$$< -\Delta v, w-u > \geq 0 \quad \forall w \in K,$$

and thus

$$< -\Delta w + \Delta u, v > \geq 0 \quad \forall w \in K. \tag{3.6}$$

Taking $w = u_0$ in (3.6) we obtain

$$< \Delta u + f, v > \geq 0. \tag{3.7}$$

Next, if $\varphi \geq 0$, φ smooth, denote by w the solution to

$$w \in H_0^1(\Omega) \quad , \quad -\Delta w = -\Delta u - \varphi.$$

One has clearly

$$\Delta w + f = \Delta u + f + \varphi \geq 0$$

so that $w \in K$. Plugging this w in (3.6) we obtain

$$< \varphi, v > \leq 0 \quad \forall \varphi \geq 0 \quad , \quad \varphi \quad \text{smooth}.$$

Thus, $v \leq 0$ q.e. in Ω. Since $\Delta u + f \geq 0$ in Ω we get

$$< \Delta u + f, v > \leq 0;$$

combining this with (3.7) we obtain

$$< \Delta u + f, v >= 0$$

and (3.4).

If now (3.4) holds, then we deduce

$$< -\Delta v, w-u >=< -\Delta w + \Delta u, v >=< -\Delta w - f, v > \geq 0 \quad \forall w \in K.$$

Thus, if $g(x,.)$ is convex :

$$\int_\Omega [g(x,w) - g(x,u)]\ dx \geq \int_\Omega g_s(x,u).(w-u)\ dx =< -\Delta v, w-u > \geq 0 \quad \forall w \in K.$$

This concludes.

As we are about to see, this function v is related to the measure $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$.

More precisely, recall that Ψ is defined by (2.1) and thus one has in particular

$$g_s(x,s) \leq 0 \quad \forall s \leq \Psi(x) \quad , \quad \text{for a.e. } \; x \in \Omega. \tag{3.8}$$

Recall also that when $\Psi(x) \geq 0$ for a.e. $x \in \Omega$ then the problems (1.17), (1.18), (1.20) are equivalent (see remark 4).

Then, we can prove :

Theorem 8 : Assume (1.4), (1.5), (2.2), (3.1), (3.2) and that

$$inf_{\Omega'} \; \Psi = m_{\Omega'} > 0 \quad \forall \Omega' \subset\subset \Omega. \tag{3.9}$$

Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimizer of (1.17). Denote by u_μ the solution to (0.2) and by v_μ the solution of (3.3) corresponding to $u = u_\mu$. Then, one has

$$v_\mu \leq 0 \quad \text{q.e. in } \; \Omega \quad , \quad v_\mu = 0 \quad \mu - \text{a.e. in } \; \Omega. \tag{3.10}$$

Conversely, when (3.5) holds, if μ is a measure in $\mathcal{M}_0(\Omega)$ which satisfies (3.10), then μ is a minimizer of (1.17).

Proof : Recall that in the case we are considering the problems (1.17), (1.18), (1.20) are equivalent (see Corollary 1 and Proposition 3). Moreover, it results from Theorem 2 that if u is a minimizer to (1.18), then

$$u > 0 \; \text{ q.e. in } \; \Omega$$

so that the measure $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ is uniquely determined (see Proposition 2).

Now, if u_μ is a minimizer of (1.20), then the inequality

$$v_\mu \leq 0 \quad \text{q.e. in } \; \Omega$$

results from Theorem 7. Moreover, if $\nu = \Delta u_\mu + f$, (3.4) gives also

$$< \nu, v_\mu >= 0.$$

Thus, $v_\mu = 0 \;\; \nu - \text{a.e. in } \; \Omega$ and thus also $\mu - \text{a.e. in } \; \Omega$ (see Proposition 2).

Conversely, assume that $\mu \in \mathcal{M}_0(\Omega)$ is such that (3.10) holds and let u_μ be the solution to (0.2). Since $v_\mu \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and $v_\mu = 0$ μ- a.e. in Ω, (0.3) gives

$$< -\Delta u_\mu, v_\mu >= \int_\Omega \nabla u_\mu \nabla v_\mu \; dx + \int_\Omega u_\mu \; v_\mu \; d\mu \; = < f, v_\mu >,$$

Thus

$$< \Delta u_\mu + f, v_\mu >= 0$$

and (u_μ, v_μ) satisfies the complementarity conditions (3.4). The result is then a consequence of Theorem 7.

Corresponding to u_0 defined by (2.6) one can associate the solution v_0 to

$$v_0 \in H_0^1(\Omega) \quad , \quad -\Delta v_0 = g_s(x, u_0) \quad \text{in} \quad \Omega. \tag{3.11}$$

Then, we can prove :

Theorem 9 : Assume that Ω is connected, Ψ satisfies (3.9) and

$$g_s(x, s) < 0 \quad \forall s < \Psi(x) \quad , \quad \text{for a.e. } x \in \Omega, \tag{3.12}$$

$$g_s(x, s) > 0 \quad \forall s > \Psi(x) \quad , \quad \text{for a.e. } x \in \Omega. \tag{3.13}$$

Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimizer of (1.17). Then μ is a Radon measure and

(i) if $\mu(\Omega) = 0$, then $u_\mu = u_0$ and $v_0 \le 0$ q.e. in Ω. Moreover, either $u_0 = \Psi$ a.e. in Ω or the set $\{u_0 < \Psi\}$ has a positive Lebesgue measure.

(ii) if $\mu(\Omega) > 0$, then either $u_\mu = \Psi$ or the sets

$$\{u_\mu < \Psi\} \quad , \quad \{u_\mu > \Psi\}$$

have both a positive Lebesgue measure.

Moreover, if the function v_μ associated to u_μ by (3.3) is continuous, then

$$u_\mu = \Psi \quad \text{a.e. in Supp } \mu, \tag{3.14}$$

if U denotes the interior of the support of μ, we have

$$\Psi \in H^1(U) \quad , \quad \Delta\Psi + f \ge 0 \quad \text{in} \quad U \quad , \quad \mu|_U = (\Delta\Psi + f)/\Psi \quad \text{in} \quad U$$

($\mu|_U$ denotes the restriction of the measure μ to U and the pointwise values of Ψ are determined according to (1.2)). The same result holds with $U = \Omega$ provided $u_\mu = \Psi$ a.e. in Ω.

Proof : The fact that μ is a Radon measure was proved in Theorem 2.

(i) If $\mu(\Omega) = 0$, then $\mu = 0$ and $u_\mu = u_0$. By (3.10) this implies $v_0 \le 0$ q.e. in Ω. If the assertion on u_0 and Ψ is false, then $u_0 \ge \Psi$ a.e. in Ω and $u_0 > \Psi$ on a set of positive Lebesgue measure. By (3.13) this implies $g_s(x, u_0) \ge 0$ a.e. in Ω and $g_s(x, u_0) \not\equiv 0$. Since

$$-\Delta v_0 = g_s(x, u_0),$$

by the strong maximum principle, $v_0 > 0$ in Ω. This contradicts $v_0(x) \leq 0$ q.e. in Ω.

(ii) Assume $\mu(\Omega) > 0$.

If our first assertion is wrong, then either

$$\begin{array}{ll} \text{a)} & u_\mu(x) \geq \Psi(x) \ \text{ a.e. in } \ \Omega \ \text{ and } \ meas(\{u_\mu > \Psi\}) > 0 \\ \text{b)} & u_\mu(x) \leq \Psi(x) \ \text{ a.e. in } \ \Omega \ \text{ and } \ meas(\{u_\mu < \Psi\}) > 0 \end{array}$$

In the first case, taking (3.13) into account we get

$$-\Delta v_\mu = g_s(x, u_\mu) \geq 0$$

and $g_s(x, u_\mu) \not\equiv 0$. By the strong maximum principle this implies $v_\mu > 0$ and a contradiction to (3.10).

In case b), by (3.12), we have

$$-\Delta v_\mu = g_s(x, u_\mu) \leq 0$$

and thus $v_\mu < 0$ q.e. in Ω, since $g_s(x, u_\mu) \not\equiv 0$. But then (see (3.10)) we would have $\mu(\Omega) = 0$ and a contradiction.

To prove (3.14), note that by (3.10), since v_μ is continuous, one has

$$v_\mu = 0 \quad \text{a.e. on } \ \mathrm{Supp}\ \mu,$$

and since $v_\mu \in H^2_{loc}(\Omega)$ (see [G.T.]) we get

$$g_s(x, u_\mu) = -\Delta v_\mu = 0 \quad \text{a.e. on } \ \mathrm{Supp}\ \mu.$$

Thus, by (3.12) and (3.13),

$$u_\mu(x) = \Psi(x) \quad \text{a.e. on } \ \mathrm{Supp}\ \mu.$$

The end of the proof is an easy consequence of Proposition 2 and Theorem 2.

Remark 7 : The function v_μ associated to u_μ by (3.3) is continuous, in particular, when

$$f \in W^{-1,s}(\Omega),\ s > n/3 \quad , \quad a_1 \in L^p(\Omega),\ p > n/2$$

(see (3.2) for the definition of a_1 and recall that $0 \leq u_\mu \leq u_0$).

4. THE RADIAL CASE

In this section we assume that

$$\Omega = B(0,R) = \{\ x \in \mathbf{R}^n \ :\ |x| < R\ \}$$

and that our data f and g are radially symmetric.

More precisely, denote by $O(n)$ the group of orthogonal transformations ρ of $\mathbf{R}^n$, i.e. the group of linear transformations such that

$$\rho^T \rho = \rho \rho^T = I.$$

(I denotes the identity of $\mathbf{R}^n$, ρ^T the transpose of ρ). Moreover, for any distribution S, denote by ρS the distribution defined by

$$< \rho S, \varphi > = < S, \varphi \circ \rho > \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{4.1}$$

($\mathcal{D}(\Omega)$ is the space of C^∞ function with compact support).
Then we assume that

$$\rho f = f \quad \forall \rho \in O(n). \tag{4.2}$$

Moreover, for every s, we suppose that there exists a function $\tilde{g}(.,s)$ on $(0,R)$ such that

$$g(x,s) = \tilde{g}(|x|,s) \quad \forall s \in \mathrm{R}, \quad \text{for a.e. } \ x \in \Omega. \tag{4.3}$$

As a consequence, the function Ψ defined in (2.1) satisfies

$$\Psi(x) = \tilde{\Psi}(|x|) \quad \text{for a.e. } \ x \in \Omega \tag{4.4}$$

for some function $\tilde{\Psi}$ defined on $(0,R)$. In what follows, for any radially symmetric function h, we will denote by $\tilde{h}$ the function defined on $(0,R)$ such that

$$h(x) = \tilde{h}(|x|).$$

It should be remarked that, if $h \in H^1(\Omega)$, then $\tilde{h} \in H^1(\epsilon, R)$ for any $\epsilon > 0$ and $\tilde{h}$ is a continuous function on $(0,R)$. Of course, it results that h itself is a continuous function on $\Omega \backslash \{0\}$.

Under these assumptions we do not know if any minimizer of (1.18) is radially symmetric. However, we have :

Proposition 5 : Assume (1.4), (1.5), (1.6), (4.2), (4.3). If

$$\tilde{g}(r,.) \text{ is convex for a.e. } r \in (0,R), \tag{4.5}$$

then there exists at least one radially symmetric minimizer to (1.18). If

$$\tilde{g}(r,.) \text{ is strictly convex for a.e. } r \in (0,R), \tag{4.6}$$

then the unique minimizer of (1.18) is radially symmetric .

Proof : First, remark that, if $u \in K^+$, then

$$\Delta(\rho u) + f = \rho(\Delta u + f) \geq 0 \qquad \forall \rho \in O(n),$$

so that $\rho u \in K^+$. Moreover, if u is a minimizer of (1.18), so does ρu, since

$$\int_\Omega g(x, \rho u(x))\, dx = \int_\Omega \tilde{g}(|x|, u(\rho^{-1}(x)))\, dx = \int_\Omega g(x, u(x))\, dx \tag{4.7}$$

(the last equality results from the change of variable $y = \rho^{-1}(x)$ in the middle integral). Now, if $g(x,s)$ is strictly convex, then, by uniqueness of the minimizer of (1.18), (see remark 3), one has

$$\rho u = u \quad \forall \rho \in O(n)$$

and thus u is radially symmetric.

If $g(x,.)$ is only supposed to be convex, and if $d\rho$ denotes the normalized Haar measure on $O(n)$, i.e. the one such that $\int_{O(n)} d\rho = 1$, one deduces from (4.7) by the Fubini Theorem and Jensen's Inequality :

$$\int_\Omega g(x,u(x))\, dx = \int_{O(n)} \int_\Omega g(x,u(x))\, dx\, d\rho = \int_{O(n)} \int_\Omega g(x, \rho u(x))\, dx\, d\rho =$$

$$\int_\Omega \int_{O(n)} g(x, \rho u(x))\, d\rho\, dx \geq \int_\Omega g(x, \int_{O(n)} \rho u\, d\rho)\, dx \geq \int_\Omega g(x, u(x))\, dx.$$

Thus $\int_{O(n)} \rho u\, d\rho$ is a radially symmetric minimizer to (1.18). This concludes the proof of the theorem. Note that the same type of result holds for K instead of K^+ provided (1.20) admits a minimizer.

In the rest of this section we would like to study more precisely the radially symmetric minimizers to (1.18). For that, consider a function $\hat{\Psi}$ satisfying :

$$\hat{\Psi} \quad \text{is radially symmetric} \quad , \quad \hat{\Psi} \in H^1(\Omega) \quad , \quad \Delta\hat{\Psi} + f \geq 0 \ \text{ in } \ \Omega. \tag{4.8}$$

Then we have :

Proposition 6 : In addition to the hypotheses of Proposition 5, assume that $\hat{\Psi}$ satisfies (4.8) and that

$$g(x,s) > g(x, \hat{\Psi}(x)) \quad \forall s < \hat{\Psi}(x) \ , \ \text{ for a.e. } \ x \ \in \Omega. \tag{4.9}$$

Then, if u denotes a radially symmetric solution to (1.18) or (1.20), there exists $r_2 \in [0, R]$ such that

$$\tilde{u}(r) \geq \tilde{\hat{\Psi}}(r) \quad \text{for} \quad 0 < r \leq r_2 \quad , \quad \tilde{u}(r) < \tilde{\hat{\Psi}}(r) \quad \text{for} \quad r_2 < r < R. \tag{4.10}$$

Proof : Denote by r_2 the largest nonnegative number such that

$$\tilde{u}(r) \geq \tilde{\hat{\Psi}}(r) \quad \forall\, r < r_2.$$

(Possibly $r_2 = 0$ or $r_2 = R$). Then one has $\tilde{u}(r) < \tilde{\hat{\Psi}}(r)$ for $r > r_2$. Indeed, otherwise the set $\{u < \hat{\Psi}\}$ would have a connected component C such that $C \subset\subset \Omega$, which contradicts Theorem 5.

Remark 8 : If u and $\hat{\Psi}$ are radially symmetric then the assumption $u,\ \hat{\Psi} \in C^0(\Omega)$ can be dropped in Proposition 4 and in Theorems 5 and 6. In fact, this hypothesis was used only to deduce that the sets $\{u < \hat{\Psi}\}$ and $\{u > \hat{\Psi}\}$ are open and to prove that $z \in H^1(\Omega)$. Since we have assumed $n \geq 2$ the same results can be obtained under the weaker hypothesis of continuity in $\Omega \backslash \{0\}$, which is always satisfied in the radially symmetric case.

Moreover, we have :

Proposition 7 : Under the assumptions of Proposition 6, suppose that $\hat{\Psi} \geq 0$ and that

$$g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s < \hat{\Psi}(x) \ , \ \text{for a.e. } x \in \Omega, \tag{4.11}$$

$$g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s > \hat{\Psi}(x) \ , \ \text{for a.e. } x \in \Omega. \tag{4.12}$$

Then, if u denotes a radially symmetric solution to (1.18) or (1.20), there exist $r_1,\ r_2$, with $0 \leq r_1 \leq r_2 \leq R$, such that

$$\tilde{u}(r) = \tilde{\hat{\Psi}}(r) \text{ for } 0 < r \leq r_1 \ , \ \tilde{u}(r) > \tilde{\hat{\Psi}}(r) \text{ for } r_1 < r < r_2 \ ,$$

$$\tilde{u}(r) < \tilde{\hat{\Psi}}(r) \text{ for } r_2 < r < R. \tag{4.13}$$

Proof : From Proposition 6 we know that there exists r_2 such that

$$\tilde{u}(r) \geq \tilde{\hat{\Psi}}(r) \quad \text{for} \quad 0 < r \leq r_2 \quad , \quad \tilde{u}(r) < \tilde{\hat{\Psi}}(r) \quad \text{for} \quad r_2 < r < R.$$

If, now, r_1 denotes the largest nonnegative number such that $\tilde{u}(r) = \tilde{\hat{\Psi}}(r)$ on $(0, r_1)$, then one has

$$\tilde{u}(r) > \tilde{\hat{\Psi}}(r) \quad \text{on} \quad (r_1, r_2).$$

Indeed, otherwise the set $\{u > \hat{\Psi}\}$ would have a connected component C such that $C \subset\subset \{u \geq \hat{\Psi}\}^0$, which contradicts Theorem 6.

Remark 9 : When $\hat{\Psi} = \Psi$, if $r_1 = r_2$, and if u is a minimizer of (1.20), then $u \leq \hat{\Psi}$ and u is the largest function satisfying

$$u \leq \hat{\Psi} \quad , \qquad \Delta u + f \ \geq 0.$$

Indeed, otherwise, there would be a function w greater than u satisfying the same inequalities (see (1.10)) and such that $w \geq u$ on a set of positive measure. This would imply that

$$\int_\Omega g(x,w)\ dx < \int_\Omega g(x,u)\ dx$$

(see (2.1)). Thus, in this case, the problem (1.20) admits a radially symmetric minimizer such that $r_1 = r_2$ if and only if this minimizer is the solution of the variational inequality

$$z \in K^{\hat{\Psi}} = \{v \in H_0^1(\Omega)\ :\ v(x) \leq \hat{\Psi}(x) \quad \text{a.e.} \quad x \in \Omega\}$$

$$< -\Delta z + f, v - z > \geq 0 \quad \forall\, v \in K^{\hat{\Psi}}.$$

In fact, under some additional assumptions, we can show that this cannot happen unless $u = u_0$ (see Theorem 10 below).

Next, we want to provide more information in the case where g is smooth.

Let Ψ be the function defined by (2.1). Assume that

$$\Psi \in H^1(\Omega) \quad , \quad \Delta\Psi + f \geq 0 \ \text{ in } \ \Omega. \tag{4.14}$$

First, remark that since $\tilde{\Psi} \in H^1(\epsilon, R)$ the limit

$$\lim_{r \to R^-} \tilde{\Psi}(r) = \tilde{\Psi}(R)$$

exists. If now this limit is 0, then, under the assumptions (3.12), (3.13) (which imply (4.11), (4.12) with Ψ instead of $\hat{\Psi}$), $u = \Psi$ is the only minimizer of (1.20). So, to avoid this trivial case still remaining in the hypotheses of Theorem 9, let us assume here

$$\Psi \ \geq \eta > 0 \quad \text{a.e. in } \ \Omega, \tag{4.15}$$

where η is a positive constant. Since, by (4.14), $\Delta\Psi$ is a (signed) Radon measure on Ω, the second derivative $\tilde{\Psi}''$ is a (signed) Radon measure on $(0, R)$. Therefore, the right and left derivatives $\tilde{\Psi}'(r^+)$ and $\tilde{\Psi}'(r^-)$ exist at any $r \in (0, R)$. The same result holds for an arbitrary radially symmetric function u of K.

Then we have :

Theorem 10 : Assume (2.2), (4.2), (4.3), (4.14), (4.15). Moreover suppose that g satisfies (1.4), (1.5), (3.1), (3.2), (3.12), (3.13). Then, if u is a radially symmetric minimizer of (1.18) one has :

There exist r_1, r_2, with $0 \leq r_1 \leq r_2 < R$, such that

$$\tilde{u}(r) = \tilde{\Psi}(r) \text{ for } 0 < r \leq r_1 \;,\quad \tilde{u}(r) > \tilde{\Psi}(r) \quad \text{for } r_1 < r < r_2 \;,$$
$$\tilde{u}(r) < \tilde{\Psi}(r) \text{ for } r_2 < r < R. \tag{4.16}$$

If $u \neq u_0$, then

$$0 < r_1 < r_2 < R. \tag{4.17}$$

In this case the measure $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ is uniquely determined by

$$\mu = \frac{\lambda|_{B(0,r_1)}}{\Psi} + \frac{f|_{\partial B(0,r_1)}}{\tilde{\Psi}(r_1)} + \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-)}{\tilde{\Psi}(r_1)}\, H^{n-1}_{\partial B(0,r_1)}, \tag{4.18}$$

where $\lambda = \Delta\Psi + f$ (which is a Radon measure by (4.14)), $H^{n-1}_{\partial B(0,r_1)}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\partial B(0,r_1)$, and $\alpha|_E$ denotes the restriction of the measure α to the measurable set E, defined as $(\alpha|_E)(B) = \alpha(B \cap E)$ for any $B \in \mathcal{B}(\Omega)$.

Remark 10 : Since the Radon measure f is radially symmetric, we have

$$f|_{\partial B(0,r_1)} = \frac{f(\partial B(0,r_1))}{H^{n-1}(\partial B(0,r_1))} H^{n-1}_{\partial B(0,r_1)}.$$

If $f \in L^2(\Omega)$ the measure μ given by (4.18) reduces to

$$\mu = \frac{\lambda|_{B(0,r_1)}}{\Psi} + \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-)}{\tilde{\Psi}(r_1)}\, H^{n-1}_{\partial B(0,r_1)}.$$

By (4.14) and (4.18) the function $u - \Psi$ is superharmonic on $A = B(0,r_2)\backslash\overline{B}(0,r_1)$ and vanishes at ∂A (assuming $u \neq u_0$). Therefore, the Hopf maximum principle implies $\tilde{u}'(r_1^+) > \tilde{\Psi}'(r_1^+)$. By (4.14), if $f \in L^2(\Omega)$ we have $\tilde{\Psi}'(r_1^+) \geq \tilde{\Psi}'(r_1^-)$, therefore $\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-) > 0$ and $\mu|_{\partial B(0,r_1)} \neq 0$. This last result holds also in the general case $f \in H^{-1}(\Omega)$, $f \geq 0$, because (4.14) implies

$$[\tilde{\Psi}'(r_1^+) - \tilde{\Psi}'(r_1^-)]H^{n-1}_{\partial B(0,r_1)} + f|_{\partial B(0,r_1)} \geq 0,$$

hence

$$\mu|_{\partial B(0,r_1)} \geq \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^+)}{\tilde{\Psi}(r_1)}\, H^{n-1}_{\partial B(0,r_1)}.$$

Proof : (4.16) results from Proposition 7.

Assume now that $u \neq u_0$.

Let μ be the measure corresponding to u. Since $u \neq u_0$, we have $\mu(\Omega) > 0$. Since $\Psi \notin H_0^1(\Omega)$ by (4.15), we have $u \neq \Psi$, thus, by Theorem 9 (ii), we get $0 \leq r_1 < r_2 < R$. Next, introduce the function v associated to u by (3.3), i.e. such that

$$v \in H_0^1(\Omega) \quad , \quad -\tilde{v}'' - \frac{n-1}{r}\tilde{v}' = \tilde{g}_s(r, \tilde{u}) \qquad \text{on} \quad (0, R).$$

Clearly, see [G.T.], $v \in H^2(\Omega)$ and $\tilde{v} \in C^1(0, R)$, the space of continuously differentiable functions on $(0, R)$. We know that $\tilde{v} \leq 0$ on $(0, R)$ by (3.10).

We claim that $\tilde{v} < 0$ on (r_1, R).

First, $\tilde{v}$ cannot vanish at $r \in (r_2, R)$. Indeed, since $u < \Psi$ in $\Omega \backslash \overline{B}(0, r_2)$, by (3.12) we have

$$-\Delta v = g_s(x, u) < 0 \quad \text{in} \quad \Omega \backslash \overline{B}(0, r_2),$$

thus the strong maximum principle implies $\tilde{v} < 0$ on (r_2, R). If now $\tilde{v}$ vanishes for some $r \in (r_1, r_2]$ then

$$v \in H_0^1(B(0, r)) \quad , \quad -\Delta v = g_s(x, u) \ \geq 0 \quad \text{in} \quad B(0, r).$$

Since $g_s(x, u) \not\equiv 0$ in $B(0, r)$ by (3.13), the strong maximum principle implies $v > 0$ in $B(0, r)$, which contradicts $v \leq 0$ q.e in Ω (see (3.10)). So, $\tilde{v} < 0$ on (r_1, R). If now $r_1 = 0$, then $v < 0$ on $\Omega \backslash \{0\}$ and by (3.10)

$$\nu = \mu = 0 \quad \text{in} \quad \Omega \backslash \{0\}$$

and thus also $\mu = 0$ in Ω. This contradicts $u \neq u_0$ and concludes the proof of (4.17).

Let us prove (4.18). We know, by Theorem 2, that $inf_{\Omega'}\ u > 0$ for every $\Omega' \subset\subset \Omega$. This implies that $1/u \in H^1_{loc}(\Omega)$ and, by Proposition 2, that

$$\mu = \frac{\Delta u + f}{u}.$$

Thus for every $\varphi \in \mathcal{D}(\Omega)$:

$$< \mu, \varphi > = < \frac{\Delta u + f}{u}, \varphi > = < f, \frac{\varphi}{u} > + < \Delta u, \frac{\varphi}{u} > = \int_\Omega \frac{\varphi}{u}\, df - \int_\Omega \nabla u . \nabla(\frac{\varphi}{u})\, dx$$

$$= \int_{\overline{B}(0,r_1)} \frac{\varphi}{u}\, df - \int_{B(0,r_1)} \nabla u . \nabla(\frac{\varphi}{u})\, dx + \int_{\Omega \backslash \overline{B}(0,r_1)} \frac{\varphi}{u}\, df - \int_{\Omega \backslash \overline{B}(0,r_1)} \nabla u . \nabla(\frac{\varphi}{u})\, dx$$

$$= \int_{\overline{B}(0,r_1)} \frac{\varphi}{\Psi}\, df - \int_{B(0,r_1)} \nabla \Psi . \nabla(\frac{\varphi}{\Psi})\, dx + \int_{\Omega \backslash \overline{B}(0,r_1)} \frac{\varphi}{u}\, df - \int_{\Omega \backslash \overline{B}(0,r_1)} \nabla u . \nabla(\frac{\varphi}{u})\, dx$$

$$= \int_{\overline{B}(0,r_1)} \frac{\varphi}{\Psi}\, df + \int_{B(0,r_1)} \frac{\varphi}{\Psi}\, d(\Delta\Psi) - \int_{\partial B(0,r_1)} \tilde{\Psi}'(r_1^-)\, \frac{\varphi}{\Psi}\, dH^{n-1}$$

$$+ \int_{\Omega\setminus\overline{B}(0,r_1)} \frac{\varphi}{u}\, d(\Delta u + f) + \int_{\partial B(0,r_1)} u'(r_1^+)\, \frac{\varphi}{\Psi}\, dH^{n-1},$$

this last equality being due to the Green formula. Since $v < 0$ on (r_1, R), by (3.10) we have $\mu = 0$ in $\Omega\setminus\overline{B}(0,r_1)$, hence

$$\Delta u + f = 0 \quad \text{in} \quad \Omega\setminus\overline{B}(0,r_1).$$

This implies

$$< \mu, \varphi >= \int_{B(0,r_1)} \frac{\varphi}{\Psi}\, d\lambda + \frac{1}{\tilde{\Psi}(r_1)} \int_{\partial B(0,r_1)} \varphi\, df + \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-)}{\tilde{\Psi}(r_1)} \int_{\partial B(0,r_1)} \varphi\, dH^{n-1}$$

and (4.18) follows.

Remark 11: Under the hypotheses of Theorem 10, if $u = u_0$, then three cases are possible:

(1) $0 = r_1 = r_2 < R$, so that $\tilde{u}_0(r) < \tilde{\Psi}(r) \quad \forall r \in (0,R)$;

(2) $0 = r_1 < r_2 < R$, so that $\tilde{u}_0(r) > \tilde{\Psi}(r)$ on $(0, r_2)$, $\tilde{u}_0(r) < \tilde{\Psi}(r)$ on (r_2, R);

(3) $0 < r_1 = r_2 < R$, so that $\tilde{u}_0(r) = \tilde{\Psi}(r)$ on $(0, r_2]$, $\tilde{u}_0(r) < \tilde{\Psi}(r)$ on (r_2, R).

The case $0 < r_1 < r_2 < R$ is excluded by the strong maximum principle, since $u_0 - \Psi$ is superharmonic in $B(0,r_2)$ and vanishes at the boundary $\partial B(0,r_2)$. The case $r_2 = R$ is excluded by the fact that $u_0 \in H_0^1(\Omega)$ and $\Psi \geq \eta > 0$ in Ω.

Examples :

Set

$$g(x,s) = (s - \Psi(x))^2 \tag{4.19}$$

where Ψ is a radially symmetric function satisfying (4.14).

Then, clearly, $g(x,.)$ is a strictly convex function for a.e. $x \in \Omega$. Thus, the problem (1.18) corresponding to this $g(x,.)$ admits a unique minimizer which is radially symmetric. Moreover, since $\Psi(x)$ is a strict minimum for $g(x,.)$, this minimizer satisfies for some $r_2 \in [0,R]$ (see Proposition 6)

$$\tilde{u}(r) \geq \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_2 \quad , \quad \tilde{u}(r) < \tilde{\Psi}(r) \quad \text{for} \quad r_2 < r < R.$$

If, in addition, $\Psi \geq 0$ in Ω, this minimizer is such that

$$\tilde{u}(r) = \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_1 \quad , \quad \tilde{u}(r) > \tilde{\Psi}(r) \quad \text{for} \quad r_1 < r < r_2$$

for some r_1 with $0 \le r_1 \le r_2$ (see Proposition 7).

If one takes the fact that $f \ge 0$ into account, a suitable Ψ is then

$$\Psi = a = constant.$$

In this case, if $a \le 0$, then the solution to (1.18) is given by $u = 0$. If $a \ge sup_\Omega u_0$, where u_0 is defined by (2.6), then the solution to (1.18) is $u = u_0$. In the other cases the above results apply. Assume, for instance, that f satisfies (2.2) and (4.2). For any $a \ge 0$ let v_a be the solution of (3.3) corresponding to $g(x,s) = (s-a)^2$ and $u = u_0$, i.e.

$$v_a \in H_0^1(\Omega) \quad , \quad -\Delta v_a = 2(u_0 - a) \quad \text{in} \quad \Omega.$$

Note that, by the strong maximum principle, we have

$$a < b \quad \Rightarrow \quad v_a > v_b \quad \text{in} \quad \Omega,$$

$$a \ge sup_\Omega u_0 \quad \Rightarrow \quad v_a < 0 \quad \text{in} \quad \Omega.$$

Let a_0 be the least constant $a \ge 0$ such that $v_a \le 0$ in Ω ($a_0 = +\infty$ if no constant a has this property). Since $f \not\equiv 0$, by the strong maximum principle, we have $u_0 > 0$ in Ω, hence, if v_0 is the solution v_a corresponding to $a = 0$, $v_0 > 0$ in Ω. This implies

$$0 < a_0 \le sup_\Omega u_0.$$

Let us prove that $a_0 < sup_\Omega u_0$ if $sup_\Omega u_0 < +\infty$. First, we observe that $\tilde{u}_0$ is decreasing on $(0, R)$ since u_0 is superharmonic in Ω. Let us fix $r_0 \in (0, R)$ such that

$$0 < \tilde{u}_0(r_0) < sup_\Omega u_0$$

and let c be a constant such that

$$0 < c < sup_\Omega u_0 - \tilde{u}_0(r_0).$$

If $\epsilon > 0$ is small enough (so that $c + \epsilon < sup_\Omega u_0 - \tilde{u}_0(r_0)$) and $a \ge sup_\Omega u_0 - \epsilon$, then

$$u_0 - a \le \epsilon \chi_{B(0,r_0)} - c\chi_{\Omega \setminus B(0,r_0)} \quad \text{in} \quad \Omega,$$

where χ_A denotes the characteristic function of A. By a comparison argument we get

$$v_a \le 2w_\epsilon \quad \text{in} \quad \Omega,$$

where w_ϵ is the solution to

$$w_\epsilon \in H_0^1(\Omega) \quad , \quad -\Delta w_\epsilon = \epsilon \chi_{B(0,r_0)} - c\chi_{\Omega \setminus B(0,r_0)} \quad \text{in} \quad \Omega,$$

and by a direct computation we obtain that $w_\epsilon \leq 0$ in Ω for $\epsilon > 0$ small enough. Therefore $v_a \leq 0$ in Ω for $a \geq sup_\Omega u_0 - \epsilon$. This gives

$$0 < a_0 < sup_\Omega u_0,$$

provided $sup_\Omega u_0 < +\infty$.

If $a \geq a_0$, then (u_0, v_a) satisfies the complementarity problem (3.4), thus $u = u_0$ is the unique minimizer of (1.20) and (1.18). If $0 < a < a_0$, then (u_0, v_0) does not satisfy (3.4), hence u_0 is not a minimizer of (1.18). In this case Theorem 10 can be applied and one has

$$0 < r_1 < r_2 < R.$$

Moreover, the unique minimizer $u = u_\mu$ of (1.18) satisfies

$$u = a \quad \text{on} \quad B(0, r_1) \quad , \quad \mu = \frac{1}{a} f|_{\overline{B}(0,r_1)} + \frac{\tilde{u}'(r_1^+)}{a} \cdot H^{n-1}_{\partial B(0,r_1)}.$$

This allows us to construct an example where the equality

$$\mu = c \, . \, H^{n-1}_{\partial B(0,r_1)}$$

holds. Indeed, u being as above, introduce the solution $\check{\Psi}$ to

$$\check{\Psi} - a \in H_0^1(B(0, r_1)) \quad , \quad -\Delta(\check{\Psi} - a) = f \quad \text{in} \quad B(0, r_1).$$

Since f is nonnegative one has clearly $\check{\Psi} - a \geq 0$ in $B(0, r_1)$. If we extend this function by 0 outside $B(0, r_1)$ - or if we still denote by $\check{\Psi}$ the extension of $\check{\Psi}$ by a outside $B(0, r_1)$ - it results from Lemma A of the Appendix that

$$\Delta\check{\Psi} + f \geq 0 \quad \text{in} \quad \Omega.$$

Now, we claim that the solution of (1.18) corresponding to

$$g(x, s) = (s - \check{\Psi}(x))^2 \tag{4.20}$$

is given by

$$\check{u} = \begin{cases} \check{\Psi} & \text{on} \quad B(0, r_1) \\ u & \text{on} \quad \Omega \backslash B(0, r_1). \end{cases}$$

Indeed, first we claim that

$$\Delta\check{u} + f \geq 0 \quad \text{in} \quad \Omega. \tag{4.21}$$

This follows from the fact that

$$\check{u} = \check{\Psi} \vee u \quad \text{in} \quad B(0, r_2),$$

hence (by (1.10))

$$\Delta \check{u} + f \geq 0 \quad \text{in} \quad B(0, r_2),$$

and

$$\Delta \check{u} + f = \Delta u + f \geq 0 \quad \text{in} \quad \Omega \backslash \overline{B}(0, r_1).$$

Next, introduce the solution $\check{v}$ to

$$\check{v} \in H_0^1(\Omega) \quad , \quad -\Delta \check{v} = g_s(x, \check{u}) = 2(\check{u} - \check{\Psi}) \quad \text{in} \quad \Omega.$$

Since

$$\check{u} - \check{\Psi} = u - a \quad \text{in} \quad \Omega,$$

$\check{v}$ coincides with the function v associated to u by (3.3) in the case $g(x, s) = (s - a)^2$. So we have

$$\check{v} \leq 0 \quad \text{q.e. in} \quad \Omega \quad , \quad \check{v} = 0 \quad \text{in} \quad B(0, r_1). \tag{4.22}$$

Moreover, since $v \in H^2(\Omega)$,

$$< \Delta \check{u} + f, \check{v} > = < \Delta \check{v}, \check{u} > + < f, \check{v} > = < \Delta v, u > + < f, v > = < \Delta u + f, v > = 0.$$

Thus, (3.4) holds and by, Theorem 7, $\check{u}$ is the minimizer of (1.18) with g given by (4.20).

Taking into account the fact that

$$\Delta \check{\Psi} + f = 0 \quad \text{in} \quad \Omega \backslash \partial B(0, r_1)$$

the measure $\check{\mu}$ associated to $\check{u}$ satisfies

$$\check{\mu} = c \, . \, H^{n-1}_{\partial B(0, r_1)}$$

with $c > 0$, since obviously $\check{u} \neq u_0$.

In the special case $n = 2$, we can describe very easily a minimizing sequence for the optimal design problem

$$\inf_A \int_\Omega (u_A - \check{\Psi})^2 \, dx \tag{4.23}$$

corresponding to the measure $\check{\mu} = c \, . \, H^1_{\partial B(0, r_1)}$ (see the introduction for the definition of u_A). We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. For every integer h, k let

$$E_h^k = \{ z \in \mathbf{C} \ : |z - r_1 e^{2\pi k i/h}| < e^{-h/c r_1} \},$$

$$E_h = \bigcup_{k=1}^{h} E_h^k \quad , \quad A_h = \Omega \backslash E_h,$$

where $i = \sqrt{-1}$. Then using the technique of Example 2.9 of [C.M.], we can prove that

$$u_{A_h} \rightharpoonup \check{u} \quad \text{weakly in} \quad H_0^1(\Omega).$$

By (0.5) this implies that (A_h) is a minimizing sequence of (4.22).

Of course, the same analysis could be performed for

$$g(x,s) = \varphi(s - \Psi(x))$$

where φ is a strictly convex function which admits a minimum at 0 and provided φ satisfies some quadratic growth condition (see (1.5)).

APPENDIX

We prove here, for the reader's convenience, a result that we have used without proof in the course of the paper.

Lemma A : Let Ω' be an open subset of Ω, $g \in H^{-1}(\Omega)$, $g \geq 0$. Let ξ be a function of $H_0^1(\Omega')$ satisfying :

$$\xi \geq 0 \quad , \quad \Delta\xi + g \geq 0 \quad \text{in} \quad \Omega'. \tag{A.1}$$

If we denote by $\tilde{\xi}$ the extension of ξ by 0 outside of Ω', then one has :

$$\Delta\tilde{\xi} + g \geq 0 \quad \text{in} \quad \Omega.$$

Proof : Set

$$K^{\tilde{\xi}} = \{v \in H_0^1(\Omega) \quad : \quad v \leq \tilde{\xi} \quad \text{a.e.} \quad \text{in} \quad \Omega\}.$$

Consider the solution w to

$$w \in K^{\tilde{\xi}} \quad , \quad < -\Delta w - g, v - w > \geq 0 \quad \forall v \in K^{\tilde{\xi}}. \tag{A.2}$$

First, we claim that $w \geq 0$ in Ω. Indeed, w is the largest subsolution to

$$-\Delta u = g$$

which belongs to $K^{\tilde{\xi}}$ (see [K.S.]). Since 0 is such a subsolution, one has $w \geq 0$ in Ω.

Next we claim that $\tilde{\xi} = w$. For that, take $v = \tilde{\xi}$ in (A.2). Thus :

$$< -\Delta w - g, \tilde{\xi} - w > \geq 0.$$

Since $0 \leq w \leq \tilde{\xi}$ in Ω, we have $\tilde{\xi} - w = 0$ outside of Ω', hence

$$< -\Delta w - g, \xi - w >_{\Omega'} \geq 0, \tag{A.3}$$

where $< \ , \ >_{\Omega'}$ denotes the duality bracket between $H^{-1}(\Omega')$ and $H_0^1(\Omega')$.

Since $\xi \geq w$ in Ω', from (A.1) we deduce

$$< -\Delta\xi - g, \xi - w >_{\Omega'} \leq 0.$$

Subtracting (A.3) we obtain

$$< -\Delta(\xi - w), \xi - w >_{\Omega'} \leq 0$$

and thus, $\xi - w = 0$ in Ω'. Since $0 \leq w \leq \tilde{\xi} = 0$ in $\Omega \backslash \Omega'$, we have proved that

$$w = \tilde{\xi} \quad \text{in } \Omega,$$

and the result follows from the fact that

$$\Delta w + g \geq 0 \quad \text{in} \quad \Omega.$$

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